Fourier Analysis

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Review.

Def. Let $f, g \in M(\mathbb{R})$. Set

$$f * g(x) := \int_{\mathbb{R}} f(x-y)\, g(y)\, dy.$$

Prop. Let $f, g \in M(\mathbb{R})$. Then

(1) $f * g = g * f$.

(2) $f * g \in M(\mathbb{R})$.

(3) $\widehat{f * g}(\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi)$.

Def. (Good kernel on $\mathbb{R}$)

A family of $(K_t)_{t\in(a,b)} \subset M(\mathbb{R})$ is said to be a good kernel on $\mathbb{R}$, as $t\to t_0$, if

(1) $$\int_{\mathbb{R}} K_t(x)\,dx = 1 \quad \text{for all } t\in(a,b).$$

(2) $$\int_{\mathbb{R}} |K_t(x)|\,dx \le M \quad \text{for all } t\in(a,b),$$

where $M>0$ is a constant.

(3) $\forall\, \delta>0$,

$$\int_{|x|>\delta} |K_t(x)|\,dx \to 0 \quad \text{as } t\to t_0.$$

Thm (Convergence thm about good kernels).

Let $(K_t)_{t\in(a,b)}$ be a good kernel on $\mathbb{R}$, as $t\to t_0$,

Let $f\in M(\mathbb{R})$. Then

$$K_t * f(x) \rightrightarrows f(x) \quad \text{on } \mathbb{R} \text{ as } t\to t_0.$$

Thm (Multiplicative formula).

Let $f, g \in \mathcal{M}(\mathbb{R})$.

Then
$$\int_{\mathbb{R}} f(x) \cdot \hat{g}(x)\, dx = \int_{\mathbb{R}} \hat{f}(x)\, g(x)\, dx.$$

Now we are ready to prove the inversion formula.

Thm (Fourier inversion formula)

Let $f \in \mathcal{M}(\mathbb{R})$. Suppose $\hat{f} \in \mathcal{M}(\mathbb{R})$.

Then
$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi$$

Pf. We first consider the case when $x = 0$.

We need to show that
$$f(0) = \int_{\mathbb{R}} \hat{f}(\xi)\, d\xi.$$

To see it, we define for $\delta > 0$,

$$g_\delta(x) = e^{-\pi\delta x^2} = e^{-\pi(x\cdot\sqrt{\delta})^2}$$

Let us take the Fourier transform of g_δ:

Recall $e^{-\pi x^2} \xrightarrow{\mathcal{F}} e^{-\pi\xi^2}$

So $e^{-\pi(\sqrt{\delta}x)^2} \xrightarrow{\mathcal{F}} \dfrac{e^{-\pi\left(\frac{\xi}{\sqrt{\delta}}\right)^2}}{\sqrt{\delta}} = \dfrac{1}{\sqrt{\delta}} e^{-\pi\frac{\xi^2}{\delta}}.$

(using $f(\delta x) \xrightarrow{\mathcal{F}} \frac{1}{\delta}\hat{f}\left(\frac{\xi}{\delta}\right)$).

Hence $$\hat{g}_\delta(\xi) = \frac{1}{\sqrt{\delta}} e^{-\pi\frac{\xi^2}{\delta}}.$$

Now set $K_\delta = \hat{g}_\delta$, $\delta > 0$.

We claim that $(K_\delta)_{\delta>0}$ is a good kernel on $\mathbb{R}$.

Check:

① $\displaystyle\int_{\mathbb{R}} K_\delta(x)\,dx = \int_{\mathbb{R}} \frac{1}{\sqrt{\delta}} e^{-\pi\frac{x^2}{\delta}}\,dx$

$\displaystyle \underset{=\!=}{\text{Letting } y=\frac{x}{\sqrt{\delta}}} \int_{-\infty}^{\infty} e^{-\pi y^2}\,dy$

$= 1.$

② $\displaystyle\int_{\mathbb{R}} |K_\delta(x)|\,dx = \int_{\mathbb{R}} K_\delta(x)\,dx = 1.$

③ $\forall\ \gamma > 0,$

$$\int_{|x|>\gamma} |K_\delta(x)|\,dx = \int_{|x|>\gamma} \frac{1}{\sqrt{\delta}} e^{-\pi\frac{x^2}{\delta}}\,dx$$

$$\underset{=\!=}{\text{Letting } y=\frac{x}{\sqrt{\delta}}} \int_{|y|>\frac{\gamma}{\sqrt{\delta}}} e^{-\pi y^2}\,dy$$

$\longrightarrow 0$ as $\delta \to 0$.

So $(K_\delta)_{\delta>0}$ is a good kernel on $\mathbb{R}$.

Hence by the convergence theorem,

$$\hat{f}(0) = \lim_{\delta \to 0} K_\delta * \hat{f}(0)$$

$$\left(K_\delta(x) = \frac{1}{\sqrt{\delta}} e^{-\pi \frac{x^2}{\delta}} \right)$$

$$= \lim_{\delta \to 0} \int_{-\infty}^{\infty} \hat{f}(x) K_\delta(-x) \, dx$$

$$= \lim_{\delta \to 0} \int_{-\infty}^{\infty} \hat{f}(x) K_\delta(x) \, dx$$

$$\left(g_\delta(x) = e^{-\pi \delta x^2} \right)$$

$$= \lim_{\delta \to 0} \int_{-\infty}^{\infty} \hat{f}(x) \widehat{g_\delta}(x) \, dx$$

by Multiplicative formula

$$= \lim_{\delta \to 0} \int_{-\infty}^{\infty} \hat{\hat{f}}(x) g_\delta(x) \, dx$$

$$= \lim_{\delta \to 0} \int_{-\infty}^{\infty} \hat{\hat{f}}(x) e^{-\pi \delta x^2} \, dx$$

Notice that $\left| \hat{\hat{f}}(x) e^{-\pi \delta x^2} \right| \leq \left| \hat{\hat{f}}(x) \right|$, $\quad |\hat{\hat{f}}| \in \mathcal{M}(\mathbb{R})$,

and

$$\lim_{\delta \to 0} \hat{\hat{f}}(x) e^{-\pi \delta x^2} = \hat{\hat{f}}(x).$$

Hence by the Dominated Convergence Thm,

$$\lim_{\delta\to 0} \int_{-\infty}^{\infty} \hat{f}(x)\, e^{-\pi\delta x^2}\, dx$$

$$= \int_{-\infty}^{\infty} \lim_{\delta\to 0} \hat{f}(x)\, e^{-\pi\delta x^2}\, dx$$

$$= \int_{-\infty}^{\infty} \hat{f}(x)\, dx.$$

This proves $\quad f(0) = \displaystyle\int_{-\infty}^{\infty} \hat{f}(x)\, dx$

$$= \int_{-\infty}^{\infty} \hat{f}(\xi)\, d\xi. \qquad (*)$$

Next we consider the general case.

Let $x_0 \in \mathbb{R}$. Define

$$f_{x_0}(x) = f(x+x_0), \quad x \in \mathbb{R}.$$

Clearly, $f_{x_0} \in \mathcal{M}(\mathbb{R})$, and $\widehat{f_{x_0}}(\xi) = \hat{f}(\xi)\cdot e^{2\pi i \xi x_0}$

$\in \mathcal{M}(\mathbb{R})$

By $(*)$,

$$f_{x_0}(0) = \int_{\mathbb{R}} \widehat{f_{x_0}}(\xi)\, d\xi.$$

Therefore

$$f(x_0) = \int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i \xi x_0}\, d\xi .$$

Since x_0 is arbitrily taken, we prove the Fourier inversion formula. $\blacksquare$.

Thm (Plancherel formula).

Let $f \in M(\mathbb{R})$. Suppose that $\hat{f} \in M(\mathbb{R})$.

Then

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, d\xi .$$

(It is an analogue of the Parseval identity).

Pf. Let $h(x) = \overline{f(-x)}$ for $x \in \mathbb{R}$.

Check: $h \in M(\mathbb{R})$.

$$\hat{h}(\xi) = \int_{-\infty}^{\infty} \overline{f(-x)}\, e^{-2\pi i \xi x}\, dx$$

$$= \int_{-\infty}^{\infty} \overline{f(-x)\, e^{2\pi i \xi x}}\, dx$$

$$= \overline{\int_{-\infty}^{\infty} f(-x)\, e^{2\pi i \xi x}\, dx}$$

Letting $y=-x$

$$\overset{\text{Letting } y=-x}{=} \overline{\int_{+\infty}^{-\infty} f(y)\, e^{-2\pi i \xi y}\, (-1)\, dy}$$

$$= \overline{\int_{-\infty}^{\infty} f(y)\, e^{-2\pi i \xi y}\, dy}$$

$$= \overline{\hat{f}(\xi)}\,.$$

Next we consider $f * h \in M(\mathbb{R})$.

Notice that

$$\widehat{f*h}(\xi) = \hat{f}(\xi) \cdot \hat{h}(\xi)$$

$$= \hat{f}(\xi) \cdot \overline{\hat{f}(\xi)}$$

$$= |\hat{f}(\xi)|^2. \qquad (z \cdot \bar{z} = |z|^2)$$

Since $\hat{f} \in M(\mathbb{R})$, so $\hat{f*h} \in M(\mathbb{R})$.

Applying the Fourier inversion formula to $f * \tilde{f}$ at $x=0$,

we obtain

$$f * \tilde{f}(0) = \int_{\mathbb{R}} \widehat{f * \tilde{f}}(\xi)\, d\xi$$

$$= \int_{\mathbb{R}} |\hat{f}(\xi)|^2\, d\xi .$$

Notice by defintion,

$$f * \tilde{f}(0) = \int_{\mathbb{R}} f(x)\, \tilde{f}(-x)\, dx$$

$$= \int_{\mathbb{R}} f(x) \cdot \overline{f(x)}\, dx$$

$$= \int_{\mathbb{R}} |f(x)|^2\, dx .$$

This proves the Plancherel formula. $\boxtimes$.

- Schwartz space $\mathcal{S}(\mathbb{R})$.

Def. Let $\mathcal{S}(\mathbb{R})$ be the collection of $f \in C^\infty(\mathbb{R})$ so that for all integers $n, \ell \geq 0$,

$$\sup_{x \in \mathbb{R}} \left| x^n f^{(\ell)}(x) \right| < \infty \qquad (**)$$

We call $\mathcal{S}(\mathbb{R})$ the Schwartz space.

Remark: $(**)$ is equivalent to

$$\left| f^{(\ell)}(x) \right| \leq \frac{C}{1 + |x|^n} \quad \text{on } \mathbb{R}.$$

Remark: ① $\mathcal{S}(\mathbb{R})$ is a vector space over $\mathbb{C}$.

If $f, g \in \mathcal{S}(\mathbb{R})$ then

$\alpha f + \beta g \in \mathcal{S}(\mathbb{R})$ for $\alpha, \beta \in \mathbb{C}$.

② If $f \in \mathcal{S}(\mathbb{R})$ then

- $x f(x) \in \mathcal{S}(\mathbb{R})$.

- $f' \in \mathcal{S}(\mathbb{R})$.

By (2), we see that if $f \in \mathcal{S}(\mathbb{R})$ then

$$P_1(x) f(x) + P_2(x) f'(x) + \cdots + P_\ell(x) f^{(\ell-1)}(x)$$

$\in \mathcal{S}(\mathbb{R})$ for all polynomials $P_1, \dots, P_\ell$.

Prop. $f \in \mathcal{S}(\mathbb{R}) \iff \hat{f} \in \mathcal{S}(\mathbb{R})$.

Pf. We only prove the direction "$\Rightarrow$".

Now let $f \in \mathcal{S}(\mathbb{R})$. We need to show that

$\forall\, n, \ell \geq 0$, $\quad \hat{f}^{(\ell)}$ exists and

$$\sup_{\xi \in \mathbb{R}} |\xi|^n \cdot |\hat{f}^{(\ell)}(\xi)| < \infty.$$

To see this, notice that

$$(-2\pi i x)^{\ell} f(x) \in \mathcal{S}(\mathbb{R}) \xrightarrow{\mathcal{F}} \hat{f}^{(\ell)}(\xi)$$

$$\frac{d^n\left((-2\pi i x)^{\ell} f(x)\right)}{dx^n} \xrightarrow{\mathcal{F}} (2\pi i \xi)^n \cdot \hat{f}^{(\ell)}(\xi)$$

$$\left(=: g(x) \right)$$

Since $g \in \mathcal{S}(\mathbb{R})$, so

$$|\hat{g}(\xi)| \leq \int_{\mathbb{R}} |g(x)|\, dx < \infty \quad \text{for all } \xi \in \mathbb{R}$$

Hence

$$\sup_{\xi \in \mathbb{R}} \left| (2\pi i \xi)^n \hat{f}^{(\ell)}(\xi) \right| \leq \int_{\mathbb{R}} |g(x)|\, dx < \infty .$$

$\blacksquare$

Statistics for Mid-term of Math 3093.

Mean = 80.4

SD = 14.7